國立中正大學
115 學年度碩士班招生考試
試 題
[第 3 節]

科目名稱	通訊理論
系所組別	通訊工程學系-通訊甲組

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

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3. 入場後於考試開始 40 分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

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1. (25%) Consider the transmission of an analog message signal $m(t)$ over a noiseless channel via an *impulse train sampler*, i.e., the transmitted signal is given by

$$x(t) = \sum_{n=-\infty}^{\infty} m(nT)\delta(t - nT),$$

where $T > 0$ denotes the sampling period and $\delta(t)$ is the Dirac delta function. Upon receiving $x(t)$, a reconstruction filter with frequency response

$$H_r(f) = \begin{cases} T, & |f| \le \frac{1}{2T}, \\ 0, & \text{otherwise}, \end{cases}$$

is used to recover the message signal; denoted by $\hat{m}(t)$ the filter output.

(a) (10%) Derive an expression for $\hat{m}(t)$ in terms of the samples $\{m(nT) : n \in \mathbb{Z}\}$.

(b) (5%) Assume that $m(t)$ is bandlimited to B(Hz) with $B < 1/(2T)$. Determine whether $m(t)$ can be perfectly reconstructed. Explain your answer.

(c) (10%) If only the signal values at the sampling instants $t = nT$ are of interest, that is, we require

$$\hat{m}(nT) = m(nT),\ \forall\, n \in \mathbb{Z},$$

is the bandlimitedness assumption on $m(t)$ necessary under this requirement?

2. (25%) Consider a conventional waveform channel $y(t) = x(t)+n(t)$, where the noise process $n(t)$ is described by a sequence of random pulse train given by

$$n(t) = \sum_{i=-\infty}^{\infty} n_i p(t - iT + \delta).$$

Here, the waveform $p(t)$ is a rectangular pulse with height $1/\sqrt{T}$ on $[0, T]$ and 0 elsewhere, the parameter $\delta \in [0, T)$ is a random time offset uniformly distributed on $[0, T)$, and the weighting coefficients n_i's are independent and identically distributed Gaussian random variables with zero-mean and variance σ_n^2. Assume that δ is independent of n_i's.

(a) (8%) Is the noise process $n(t)$ wide-sense stationary (WSS)? Explain your answer.

(b) (6%) Describe the limiting behavior of the autocorrelation function and power spectral density of $n(t)$ as $T \to 0$.

(c) (4%) Let $w(t)$ be the output of the ideal low-pass filter of bandwidth B(Hz) and height 1 with input $n(t)$. Find the output variance of $w(t)$ in the limit as $T \to 0$.

(d) (7%) As $T \to 0$, the noise process $n(t)$ becomes Gaussian since any finite number of samples of $n(t)$ are jointly Gaussian. Discuss whether $w(t)$ defined in part (c) is also a Gaussian process and whether it is WSS. Explain your answer.

3. (20%) Consider that a single-tone message signal $m(t) = A_m \cos(2\pi f_m t)$ is used to generate the following modulated signal with carrier amplitude A_c and frequency f_c ($\gg f_m$):

$$s(t) = \frac{1}{2} a A_m A_c \cos[2\pi(f_c + f_m)t] + \frac{1}{2}(1-a) A_m A_c \cos[2\pi(f_c - f_m)t],$$

where $a \in [0,1]$ denotes the attenuation factor applied to the upper side frequency.

(a) (4%) Find the in-phase and quadrature components of $s(t)$.

(b) (4%) Identify the modulation type represented by $s(t)$ for $a = 0$, $a = 0.5$, $a = 1$, and for other values of a.

(c) (8%) Assume that the signal $s(t)$ is, plus the carrier $A_c \cos(2\pi f_c t)$, is sent and passed through an envelope detector. In the absence of channel noise, determine the envelope distortion introduced by the quadrature component in the detector output.

(d) (4%) Based on your result in part (c), determine the values of a for which the distortion is eliminated and the values for which it is maximized. Justify your answer.

4. (30%) Consider the discrete-time channel

$$\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x} + \boldsymbol{n},$$

where "$*$" is the convolution operator, $\boldsymbol{h} = [1, -1, 1]$ denotes the channel impulse response, $\boldsymbol{x} = [x_1, x_2]$ represents the transmitted vector with two independent and equiprobable signals selected from the set $\{+1, -1\}$, and the noise vector $\boldsymbol{n}$ consists of independent and identically distributed Gaussian random variables with zero-mean and variance σ_n^2. Assume full linear convolution with zero padding. Answer the following questions:

(a) (2%) Express $\boldsymbol{y}$ in matrix form $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$ by specifying the channel matrix $\boldsymbol{H}$.

(b) (5%) Based on the representation in part (a), derive the maximum-likelihood (ML) decision rule for estimating x_1 from $\boldsymbol{y}$, given $\boldsymbol{H}$ and the noise statistics.

(c) (5%) Assume that σ_n^2 is sufficiently small, derive an approximation of the ML decoding error probability for x_1 under the decision rule obtained in part (b). Express your result in terms of the Q-function, where $Q(t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$.

(d) (5%) In the absence of noise, design a linear left-inverse system $\boldsymbol{W}$ that minimizes the mean-squared error $\mathbb{E}_{\boldsymbol{x}}\left[\|\boldsymbol{W}\boldsymbol{y} - \boldsymbol{x}\|^2\right]$, where the expectation is taken over all equiprobable transmitted vectors $\boldsymbol{x}$ and $\|\cdot\|$ denotes the Euclidean norm.

(e) (5%) Let $\tilde{\boldsymbol{y}} = \boldsymbol{W}\boldsymbol{y}$ denote the output vector of the inverse system in part (d). Determine the probability distribution of the effective noise vector $\tilde{\boldsymbol{n}} = \boldsymbol{W}\boldsymbol{n}$.

(f) (5%) Suppose that the optimal decoding of x_1 is derived based on the scalar observation $\tilde{y}_1 = x_1 + \tilde{n}_1$. What is the decoding error probability for sufficiently small σ_n^2? Express your result in terms of the Q-function defined in part (c).

(g) (3%) Even though both decoders are optimal under their respective models, one based on $\boldsymbol{y}$ and the other based on $\tilde{\boldsymbol{y}}$, their performance is not the same. Explain why.

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試 題

[第 4 節]

科目名稱	線性代數與機率
系所組別	通訊工程學系-通訊甲組

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國立中正大學 115 學年度碩士班招生考試試題

科目名稱：線性代數與機率　　　　本科目共 2 頁　第 1 頁

系所組別：通訊工程學系-通訊甲組

1. Consider the following system of linear equations:

$$x_1 + x_2 + x_3 = 2$$
$$x_2 + 2x_3 = 1$$
$$(a^2 - 9)x_3 = a - 2$$

Based on the parameter a, answer the following questions and explain your reasoning:

(a) (3%) For which condition(s) of a does the system have exactly one solution?

(b) (3%) For which value(s) of a does the system have zero solutions?

(c) (3%) For which value(s) of a does the system have infinitely many solutions?

2. Let $\mathbf{W}$ be the plane with equation $x + 2y - 3z = 0$.

(a) (3%) Find a basis for $\mathbf{W}$.

(b) (6%) Find the standard matrix for the orthogonal projection onto $\mathbf{W}$.

(c) (3%) Use the matrix obtained in (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ onto $\mathbf{W}$.

(d) (3%) Find the distance between the point $P_0(-1, 1, -1)$ and the plane $\mathbf{W}$, and check your result.

3. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(a) (6%) Find the transition matrix $P_{B,B'}$ from B to B'.

(b) (6%) Find the transition matrix $P_{B',B}$ from B' to B.

(c) (2%) Compute the coordinate matrix $[\mathbf{w}]_{B'}$ where $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

4.

$$M = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) (6%) Find a diagonalization $M = PDP^{-1}$ and use it to compute the matrix power M^7.

(b) (6%) Compute a QR-decomposition of M; that is, find a matrix Q whose columns form an orthonormal set and an upper triangular matrix R such that $M = QR$.

5. The number N of packet arrivals in t seconds at a multiplexer is a Poisson random variable with arrival rate of λ packets per second.
   (a) (5%) Find the mean of N in t seconds.
   (b) (5%) Find the probability that there are no packet arrivals in t seconds.
   (c) (5%) Let Z be the time until the first packet arrival. What is the probability that $Z \leq t$?
   (d) (5%) Find the mean of packet interarrival times.

6. Let $Y = e^X$, where X is a random variable.
   (a) (5%) Find the cumulative distribution function (cdf) of Y in terms of the cdf of X.
   (b) (4%) Find the probability density function (pdf) of Y in terms of the pdf of X.
   (c) (5%) Find the pdf of Y when X is a Gaussian random variable with mean m and variance σ^2.
   (d) (8%) Now let $Y = e^{-\lambda X}$ rather than $Y = e^X$. Find the pdf of Y when X is an exponential random variable with parameter λ. What random variable is Y?

7. Let X be a standard normal random variable.
   (a) (5%) Find the characteristic function (c.f.) of X.
   (b) (3%) Find the c.f. of $Y = \sigma X + m$, where $\sigma \neq 0$ and m are real numbers.